Control aspects of linear discrete time-varying systems

Engwerda, J.C.

Published in:
International Journal of Control

Publication date:
1988

Citation for published version (APA):
Engwerda, J. C. (1988). Control aspects of linear discrete time-varying systems. *International Journal of Control*, *48*(4), 1631-1658.

INT. J. CONTROL, 1988, VOL. 48, NO. 4, 1631–1658

Control aspects of linear discrete time-varying systems

J. C. ENGWERDA†

Results for time-invariant linear systems concerning controllability, reachability, output controllability, stabilizability, and target path controllability are extended to time-varying systems. For the extensions which are already known in the literature elementary proofs are given. Moreover, necessary and sufficient conditions are given for the (strong) admissibility of a target path.

1. Introduction

The exact tracking of a target path has attracted much attention both in control theory and economics. After the pioneering work of Tinbergen (1952), Brockett and Mesarovic (1965) formulated necessary and sufficient conditions for target path controllability in linear continuous time-invariant systems. Since then several papers have appeared on this subject (Albrecht *et al.* 1986, Aoki 1975, Aoki and Canzoneri 1979, Preston 1974, Preston and Sieper 1977, Preston and Pagan 1982, Sain and Massey 1969, Wohltmann 1981), while more recently Wohltmann (1985) and Grasse (1986) have extended the results to continuous time-varying systems.

This paper is concerned with discrete time-varying systems. After introducing some notation and establishing some elementary lemmas in § 2, we treat target point problems in § 3. Although these problems have been treated before (see e.g. Meditch 1969, Theorems 2.2 and 2.4; Ludyck 1981, Theorems 2.2 and 2.4; and Kwakernaak and Sivan 1972, Theorem 6.7), either the solutions given were incomplete, or the proofs were false. So for the sake of completeness we give results here for controllability, stabilizability, output, and reachability problems, together with elementary proofs.

In § 4 we turn to target path controllability problems. The aim is to track a certain predescribed target path for a given time interval. A preliminary question is whether there exists a number k such that any target path can be tracked for a period of length k.

Subsequent questions concern the maximum of such numbers k and the reaction time, i.e. the time needed to reach the target path (also known as the policy lead question). In the work by Preston and Pagan (1982), and Wohltmann (1981) this problem has been solved for time-invariant systems. Here we derive an algebraic algorithm for time-varying systems. This section is followed by a section in which the decoupling problem is solved.

In § 6 we take another point of view. Given a system we give a characterization of all target paths which can be tracked. Section 7 gives a characterization of target paths which can be tracked asymptotically. The paper ends with a conclusion.

Received 2 November 1987. Revised 10 December 1987.

† Department of Mathematics and Computing Science, University of Technology, Den Dolech 2, P.O. Box 513, 5600 MB Eindhoven, The Netherlands.

2. Preliminaries

In this section we consider a system described by the following linear time-varying discrete time recurrence equation:

$$\left.\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k), \quad x(k_0) = x \\ y(k) &= C(k)x(k) \end{aligned}\right\} \Sigma$$

Here $x(k) \in \mathbb{R}^n$ is the state of the system, $u(k) \in \mathbb{R}^m$ is the applied control, and $y(k) \in \mathbb{R}^r$ is the output at time k.

We shall use the following notation: $y^*(i)$ denotes a reference value for variable y at time i; $y^T(i)$ denotes the transpose of $y(i)$;

$$y[k, l] := (y^T(k), \ldots, y^T(l))^T$$

$$y[k, \cdot] := (y^T(k), y^T(k+1), \ldots)^T$$

Im A denotes the image of the mapping defined by matrix A, Ker A denotes its kernel;

$$A(k+i, k) := A(k+i) \times \ldots \times A(k), \quad \text{for } i \geq 0$$

$$\begin{aligned} S[N+i, N] := [&B(N+i) | A(N+i)B(N+i-1) | \ldots | \\ &A(N+i, N+1)B(N)], \quad \text{for } i \geq 1 \end{aligned}$$

$$\begin{aligned} W[N, N+i] := [&C^T(N) | \ldots | \{C(N+i) \\ &\times A(N+i-1, N)\}^T] \quad \text{for } i \geq 1 \end{aligned}$$

$x(k, k_0, x_0, u)$ is the state of the system at time k resulting from the initial state x_0 at time k_0 when the input $u[k_0, k-1]$ is applied; and

$$y(k, k_0, x_0, u) := C(k)x(k, k_0, x_0, u)$$

Throughout this paper norms are used. As well for vectors as matrices, the norm we use is assumed to be the euclidian norm, i.e.

$$\text{if } y = (y_1, \ldots y_r) \quad \text{then} \quad \|y\|^2 = \sum_{i=1}^{r} y_i^2$$

We proceed with giving a number of elementary lemmas which are used in the forthcoming sections. Lemma 1 gives a necessary condition on the additive noise component in a recursive linear error equation when convergence of error is desired. In general this condition is not sufficient.

Lemma 1

Let $\|A(k)\| \leq c$ for all k, and let $\{e(k)\}$ satisfy

$$e(k+1) = A(k)e(k) + v(k)$$

Then $e(k) \to 0$ implies $v(k) \to 0$.

Proof

The proof follows directly from the equality $v(k) = e(k+1) - A(k)e(k)$. □

Lemma 2 gives necessary and sufficient conditions for the solvability of a linear equation.

Lemma 2

Let B be an $n \times m$ matrix. Then the equation $Bu = y$ is solvable if and only if (iff) rank $[B|y] = \text{rank } B$. Moreover, the solution is uniquely determined iff rank $B = m$. Then the solution is given by

$$u = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}y$$

Proof

A proof is given by Lancaster and Tismenetsky (1985, Chap. 3.10).

Note that if matrix B in this lemma is not full column rank there always exists a transformation S in the input space U such that BS equals $[B'|0]$, where now matrix B' is full column rank. So if in this case the problem is solvable, then there exist infinitely many solutions. Lemma 3 has a more set theoretic background. It tells us that $\mathbb{R}^n$ cannot be covered by a countable set of linear subspaces, unless one of them equals $\mathbb{R}^n$.

Lemma 3

Let I be an index set and $S(i)$ and S be linear subspaces of $\mathbb{R}^n$ with dimension $S(i) <$ dimension S. Then $\bigcup_{i \in I} S(i) = S$ implies that I is uncountable.

Lemma 3 is used in the next section to derive a result for output controllability. The next three lemmas are used in § 4 to deduce an algorithm for target path controllability. Since the reader may not be familiar with the results short proofs are provided.

Lemma 4

Let $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{p \times m}$. Then $\begin{bmatrix} A \\ B \end{bmatrix}$ is full row rank iff the following hold:

(*a*) Im $A = \mathbb{R}^n$, and

(*b*) B Ker $A = \mathbb{R}^p$

or equivalently

(*a*) Im $B = \mathbb{R}^p$, and

(*b*) A Ker $B = \mathbb{R}^n$.

Proof

We first note that $m \geqslant n + p$ is a necessary condition. Now consider the following decomposition of $\mathbb{R}^m : \mathbb{R}^m = X \oplus \text{Ker } A$. With respect to this basis, the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ has the following structure:

$$\begin{bmatrix} A' & 0 \\ B' & B'' \end{bmatrix}$$

where A' is full column rank. Therefore $\begin{bmatrix} A \\ B \end{bmatrix}$ is full row rank iff A' is invertible and B' is full row rank. □

Corollary 1

Let $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{n \times q}$, $C \in \mathbb{R}^{p \times q}$. Then $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ is full row rank iff the following hold:

(a) Im $C = \mathbb{R}^p$;

(b) Im $A + B$ Ker $C = \mathbb{R}^n$.

Lemma 5

Let s be a homomorphism from $V \to W$, and L any linear subspace of V. Then

$$s^{-1}(s(L)) = L + \text{Ker } s$$

where s^{-1} means inverse mapping.

Corollary 2

Let $A \in \mathbb{R}^{n \times m}$ and V be a linear subspace of $\mathbb{R}^m$. Then $AV = \mathbb{R}^n$ is equivalent to the following:

(a) Ker $A + V = \mathbb{R}^m$; and

(b) A is full row rank.

Lemma 6

$$\text{Ker } C \cap A \text{ Ker } B = A \text{ Ker} \begin{bmatrix} CA \\ B \end{bmatrix}$$

Proof

$x \in \text{Ker } C \cap A \text{ Ker } B \Leftrightarrow$

there exists a vector b such that $Cx = 0$, $x = Ab$ and $Bb = 0 \Leftrightarrow$

there exists a $b \in \text{Ker} \begin{bmatrix} CA \\ B \end{bmatrix}$ such that $x = Ab \Leftrightarrow$

$x \in A \text{ Ker} \begin{bmatrix} CA \\ B \end{bmatrix}$ □

3. Point controllability problems

In this section several point controllability problems are analysed. To help the reader to see clearly the differences between all the concepts that are treated, we place the most important ones together in one definition. Once we have introduced this basic definition, subspaces which are closely related to these concepts are defined and remarks are made about these definitions.

Definition 1

The initial state x of the system Σ is said to be as follows:

(a) asymptotically stable at k_0: if $\lim_{k \to \infty} x(k, k_0, x, 0) = 0$;

(*b*) stabilizable at k_0: if $\exists u[k_0, \cdot]$ such that $\lim_{k\to\infty} x(k, k_0, x, u) = 0$;

(*c*) zero controllable at k_0: if $\exists u[k_0, N-1]$ with $k_0 < N < \infty$ such that $x(N, k_0, x, u) = 0$.

The state x is said to be reachable at k_0 from zero if $\exists u[N, k_0 - 1]$ with $-\infty < N < k_0$ such that $x(k_0, N, 0, u) = x$, while the output y is said to be output controllable from zero at k_0 if $\exists u[k_0, N-1]$ with $k_0 < N < \infty$ such that $y(N, k_0, 0, u) = y$.

Now let L be a linear subspace of $\mathbb{R}^n$ (resp. $\mathbb{R}^r$). Then Σ is called L-zero controllable at k_0 if all states $x \in L$ are zero controllable at k_0. Completely analogous are L-asymptotic stability, L-stabilizability, L-reachability, and L-output controllability of Σ defined. For each of these concepts there is a maximal subspace L having this property. For these maximal subspaces we introduce some terminology which stems from time-invariant systems.

In the sequel the subspace consisting of all asymptotically stable states at k_0 is denoted by $X^-(A(k_0, \cdot))$. The stabilizability subspace at k_0 is the subspace consisting of all initial states at k_0 which are stabilizable and is abbreviated by $X_{\text{stab}}(k_0)$. Finally, the subspaces of all zero controllable states at k_0 and reachable states from zero at k_0 are denoted respectively by Z_{k_0} and R_{k_0}. That all the spaces defined here are indeed linear subspaces is easily shown. In case the maximal subspaces equal $\mathbb{R}^n$ (resp. $\mathbb{R}^r$) we talk respectively about asymptotic stability, stabilizability, zero controllability, reachability, and output controllability of Σ at k_0.

Remark 1

In the last defined system properties the subscript 'from zero' is dropped for reachability and output controllability. This is due to the fact that reachability respectively output controllability from zero implies reachability respectively output controllability from any initial state. By considering the system $x(k) = 0$ for all k, we see that this implication does not hold for zero controllability.

Remark 2

The concepts of zero controllability and reachability are dual. This property is used in the proof of a result about reachability.

Remark 3

In the literature the concept of output controllability of the system at k_0 to a prespecified target y^* is often encountered. Usually this is defined as the property of Σ that for any initial state x at k_0 there exists a finite control sequence $u[k_0, N-1]$ such that $y(N, k_0, x, u) = y^*$. Using this concept, then output controllability of Σ at k_0 can also be defined. We prefer, however, to define output controllability as the property that the output can be controlled towards any reference point, given the initial state of Σ.

It has already been noted in the introduction that many proofs have been given in the past concerning results for zero controllability and reachability which have proved to be wrong for the most general system, as considered here. For this reason we will not only state here the results but also give elementary proofs of them. The first result is about zero controllability.

Theorem 1

Σ is L-zero controllable at k_0 iff there exists an integer $M > k_0$ such that $A(M-1, k_0)L \subset \operatorname{Im} S[M-1, k_0]$.

Proof

Clearly $A(M-1, k_0)L \subset \operatorname{Im} S[M-1, k_0]$ is equivalent to: for all $l \in L$, $\exists u[M-1, k_0]$ such that

$$0 = A(M-1, k_0)l + S[M-1, k_0]u[M-1, k_0]$$

From this the sufficiency of the condition is clear.

To prove the necessity of the condition we let $e_1, \ldots, e_k$ be a basis for L. Assume that $u_i[k_0, N_i - 1]$ steers e_i to zero, and that $M = \max N_i$. If $N_i < M$ for some index i then we define a new extended control sequence as follows:

$$u_i[k_0, M-1] := (u_i^{\mathrm{T}}[k_0, N-1], 0, \ldots, 0)^{\mathrm{T}}$$

Let l be any element of L, say $l = \sum_{i=1}^{k} \alpha_i e_i$. Then the control sequence $\sum_{i=1}^{k} \alpha_i u_i[k_0, M-1]$ steers l to zero at M. So for $l \in L$ the equation

$$0 = A(M-1, k_0)l + S[M-1, k_0]u[M-1, k_0]$$

holds. From this the necessity of the inclusion is clear, too. □

Note that as a special case of Theorem 1 we obtain that Σ is zero controllable iff there exists an integer $M > k_0$ such that $\operatorname{Im} A(M-1, k_0) \subset \operatorname{Im} S[M-1, k_0]$.

Our second result concerns reachability. The proof of it is a complete dualization of the proof of Theorem 1.

Theorem 2

Σ is L-reachable from zero at k_0 iff there exists an integer $M < k_0$ such that $L \subset \operatorname{Im} S[k_0 - 1, M]$.

Proof

It is easily shown that the condition is sufficient. To prove the necessity of the condition let $e_1, \ldots, e_k$ be a basis for L again. Then for each e_i there exists an input sequence $u_i[N_i, k_0 - 1]$ such that this input sequence steers the state from zero at N_i to e_i at k_0.

Let M be the minimum of N_i, for $i = 1, \ldots, k$. If $N_i > M$ for some index i then we define the new control sequence $u_i[M, k_0 - 1]$ as follows:

$$u_i[M, k_0 - 1] := (0, \ldots, 0, u_i^{\mathrm{T}}[N_i, k_0 - 1])^{\mathrm{T}}$$

Now let l be any element of L, say $l = \sum_{i=1}^{k} \alpha_i e_i$. Then the control sequence $\sum_{i=1}^{k} \alpha_i u_i[M, k_0 - 1]$ steers the initial state of the system from zero to l at k_0. In other words there exists an integer M such that any $l \in L$ can be reached at k_0 from zero at M. So for any $l \in L$ there exists an input sequence $u[M, k_0 - 1]$ such that the following equation holds:

$$l = S[k_0 - 1, M]u[k_0 - 1, M]$$

which proves the theorem. □

By taking L equal to $\mathbb{R}^n$ we obtain that Σ is reachable at k_0 iff there exists an integer $M < k_0$ such that rank $S[k_0 - 1, M] = n$.

The next item we discuss is output controllability. A special case concerning this subject has been stated by Kwakernaak and Sivan (1972, Theorem 6.7). However, no proof has been provided there. We give here a proof, though not constructive, of this generalization from his concept of complete controllability. To give a constructive proof is difficult since an output which can be obtained at time k is, in general, unobtainable at time $k + 1$.

Theorem 3

Σ is L-output controllable from zero at k_0 iff there exists an integer $M > k_0$ such that

$$L \subset \operatorname{Im} C(M)\, S[M - 1, k_0]$$

Proof

The sufficiency of the condition is again easily shown. The necessity of the condition is proved by contradiction. Assume that for any integer $M > k_0$, $L \not\subset \operatorname{Im} C(M)\, S[M - 1, k_0]$. Then $L \cap \operatorname{Im} C(M)\, S[M - 1, k_0]$ is for any M a linear subspace with dimension smaller than the dimension of L. Since any output $y \in L$ can be obtained from zero at k_0 we know that the collection $\bigcup_{i \in I} L \cap \operatorname{Im} C(M)\, S[i - 1, k_0]$ covers L where $I = \{k_0, k_0 + 1, \ldots\}$. So we have a countable collection of subspaces, all with lower dimension than the dimension of L, which cover L. This clearly contradicts Lemma 3. □

A special case of Theorem 3 is again obtained by taking L equal to $\mathbb{R}^r$. The theorem states then that Σ is output controllable from zero iff there exists an integer $M > k_0$ such that rank $C(M)\, S[M - 1, k_0] = r$. Another interesting aspect is that the rank condition given in Theorem 3 is equivalent to the following statement: there exists an integer $M > k_0$ such that for all $y \in L$ an input sequence $u[k_0, M - 1]$ exists which steers the output from the inial state $x(k_0) = 0$ to y at M. This is a result which, due to the linearity of Σ, also holds for L-zero controllability and L-reachability. So Σ has one of these properties iff there exists a uniform time M at which for each $l \in L$ this property holds. In other words in the definition of these concepts the quantors $\forall l \in L$, $\exists N$ such that etc. may be interchanged.

The last concept treated in this section is stabilizability. For time-invariant systems we know that the stabilizability subspace is given by

$$X^-(A) + Z \tag{1}$$

For these systems it can be shown that if a system is stabilizable at k_0 then there exists a control such that from time $k_0 + n$ on, a control is no longer needed to obtain convergence of all states towards zero. This property does not hold any longer for time-varying systems. A simple example illustrates this phenomenon.

Example 1

Take

$$A(k)=\begin{bmatrix}2 & 0\\ 0 & \frac{1}{2}\end{bmatrix},\quad B(k)=\begin{bmatrix}1\\ 0\end{bmatrix},\quad C(k)=I,\quad k\neq 100, 200, \ldots$$

$$A(k)=\begin{bmatrix}2 & \dfrac{1}{k}\\ 0 & \frac{1}{2}\end{bmatrix},\quad B(k)=\begin{bmatrix}1\\ 0\end{bmatrix},\quad C(k)=I,\quad k=100, 200, \ldots$$

This system is stabilizable. However, at any point $(100\times k+1)$ in time we must control the system in order to achieve this property. Moreover we see that $X^{-}(A(k_0,\cdot))=0$ and $Z_{k_0}=\begin{bmatrix}1\\ 0\end{bmatrix}$. So the property (1) ceases to hold.

In the rest of this section we embed the stabilizability subspace in a subspace from which one expects at first glance that it equals the stabilizability subspace. Though this is not the case in general it turns out that for a very broad class of systems equality is obtained. The advantage of this new introduced subspace—the potential stabilizibility subspace—is that it can be calculated as a stability subspace. The introduction of this subspace requires a state space decomposition. This state space decomposition is based on the property that the time-dependent reachability subspace is $A(k)$ invariant—i.e. $A(k)R_k\subset R_{k+1}$. This property is now proved.

Lemma 7

$$A(k)R_k+\operatorname{Im}B(k)=R_{k+1}$$

Proof

It is obvious that if x is reachable at k, $A(k)x$ is reachable at $k+1$—take $u(k)=0$. Furthermore, any element in the image of $B(k)$ is reachable at $k+1$ (see Definition 1). So one inclusion is clear.

To prove the other inclusion we use the fact that for any k there exists an integer $N(k)$ such that

$$R_k=\operatorname{Im}S[k-1,k-1-N(k)]$$

By definition of R_k this implies that

$$\operatorname{Im}S[k-1,k-1-N]=R_k$$

for any $N>N(k)$. Now let k be a given number. We make a distinction between two cases, namely, $N(k+1)>N(k)$ and $N(k+1)<N(k)$. First consider the case $N(k+1)>N(k)$. Then

$$\begin{aligned}A(k)R_k+\operatorname{Im}B(k)&=A(k)\operatorname{Im}S[k-1,k-1-N(k)]+\operatorname{Im}B(k)\\&=A(k)\operatorname{Im}S[k-1,k-1-N(k+1)]+\operatorname{Im}B(k)\\&=\operatorname{Im}S[k,k-N(k+1)]\end{aligned}$$

which proves one case. On the other hand, if $N(k+1) < N(k)$, we have

$$\begin{aligned} R_{k+1} &= \operatorname{Im} S[k, k - N(k+1)] \\ &= A(k) \operatorname{Im} S[k-1, k-1-N(k+1)] \\ &\subset A(k) \operatorname{Im} S[k-1, k-1-N(k)] \\ &= A(k) R_k + \operatorname{Im} B(k) \end{aligned}$$

which completes the proof. □

The required state space decomposition results immediately from Lemma 7. A proof of it has been given by Ludyck (1981, Theorem 6.1).

Denote the dimension of the reachable subspace at time k, R_k, by r_k and let $0_{p,q}$ denote a zero matrix with p rows and q columns.

Corollary 3

Let $\mathbb{R}^n = R_k \oplus X_k$. With respect to a basis adapted to this decomposition Σ is described by the following recurrence equation:

$$\begin{bmatrix} x'_1(k+1) \\ x'_2(k+1) \end{bmatrix} = \begin{bmatrix} A'_{11}(k) \,|\, A'_{12}(k) \\ 0_{n-r_{k+1}, r_k} \,|\, A'_{22}(k) \end{bmatrix} \begin{bmatrix} x'_1(k) \\ x'_2(k) \end{bmatrix} + \begin{bmatrix} B'_1(k) \\ 0_{n-r_{k+1}, m} \end{bmatrix} u(k)$$

$$y(k) = (C'_1(k) \,|\, C'_2(k)) \begin{bmatrix} x'_1(k) \\ x'_2(k) \end{bmatrix}$$

where the subsystem $x'_1(k+1) = A'_{11}(k)x'_1(k) + B'_1(k)u(k)$ is reachable at any time k, the state space dimension of $x'_1(k)$ is r_k and that of $x'_2(k)$ is $n - r_k$.

Note that this state space decomposition can be obtained by a state transformation $x'(k) = T(k)x(k)$.

We now define the potential stabilizability subspace at k_0. Therefore, we reconsider Σ for $k > k_0$, assuming that $A(k)$ and $B(k)$ are zero for all $k < k_0$. Furthermore, we introduce $A'(k) := A(k) \bmod R_k$. This is a mapping from $\mathbb{R}^n \bmod R_k \to \mathbb{R}^n \bmod R_{k+1}$ which, due to Lemma 7, makes sense. Note that for $k = k_0$ we have that $A'(k_0) = A(k_0)$.

Definition 2

The potential stabilizability subspace at k_0 is defined as follows:

$$X^-(k_0) := \{x \in \mathbb{R}^n \,|\, x(i+1) = A'(i)x(i) \to 0, x(k_0) = x\}$$

To motivate the study of this potential stabilizability subspace we first characterize this subspace in two examples. The first example we consider is the autonomous system, i.e. $B(k) = 0$ for all k. Here $R_k = 0$ for all k. Consequently the potential stabilizability subspace coincides with the stability and the stabilizability subspaces. The second example concerns time-invariant systems, i.e. $A(k) = A$, and $B(k) = B$ for all k. Here the potential stabilizability subspace coincides with the stabilizability subspace, as is shown later on in this section. These two examples suggest that the potential stabilizability subspace is closely related to the stabilizability subspace. The exact relation is revealed in Theorems 4 *a* and 4 *b*. The first theorem gives a justification of the chosen name for the potential stabilizability subspace.

Theorem 4 *a*

$$X_{\text{stab}}(k_0) \subset X^-(k_0)$$

Proof

Let x be an element of the stabilizability subspace at k_0. By definition there exists then a control sequence $u[k_0, \cdot]$ such that in

$$x(k+1) = A(k)x(k) + B(k)u(k)$$

$x(k_0) = x$, $\lim x(k, k_0, x, u) = 0$. However, this is equivalent to the existence of a control sequence $u'[k_0, \cdot]$ such that, in the transformed system

$$x'(k+1) = A'(k)x'(k) + B'(k)u'(k)$$

$x'(k_0) = x$, $\lim x'(k, k_0, x, u) = 0$. Take the transformation now conforming to Corollary 3. Then we observe that

$$x'_1(k+1) = A'_{11}(k)x'_1(k) + A'_{12}(k)x'_2(k) + B'_1(k)u'(k)$$
$$x'_2(k+1) = A'_{22}(k)x'_2(k), \quad x'(k_0) = x$$

where the state space dimension of $x'_1(k)$ is time-varying and may even become zero. So the matrices $A'_{11}(k)$ and $B'_1(k)$ may disappear (as happens e.g. already for $k = k_0$). Anyway, it is clear that if x is stabilizable then necessarily in the above decomposed system

$$x'_2(k+1) = A'_{22}(k)x'_2(k)$$

with $x'(k_0) = x$, converges to zero when k tends to infinity. This implies that x is an element of the potential stabilizability space, which had to be proved. □

To prove the converse statement we have to construct a control sequence which steers incoming exogenous influences arising from the second state component (see Corollary 3) to zero in the first state component. Systems which have this property will be called smooth controllable in the sequel. A proper definition is given below. However, even in the case where the whole second state component of the system converges to zero it is, in general, not possible to construct such a sequence which results in the convergence of $x(k, k_0, x, u)$ to zero. An example of such a system is Example 1 with matrix $B(k)$ replaced by $\begin{bmatrix} (\frac{1}{2})^{10k} \\ 0 \end{bmatrix}$. So in general the stabilizability subspace does not coincide with the potential stabilizability subspace. The main problem that remains to be solved is to find conditions from which easily smooth controllability of a system can be concluded. A sufficient condition is when periodically these incoming exogenous influences can be steered to zero by means of a bounded input sequence. If in each period, at least once, the inverse of the controllability gramian of the reachable subsystem remains uniform bounded this condition is satisfied. This idea is reflected in the next definition of periodic smooth controllability.

Definition 3

Σ is called smooth controllable if $X_{\text{stab}}(k_0) = X^-(k_0)$.

Definition 4

Let $S'[k, k-N]$ be a matrix obtained from Corollary 3 in the following way:

$$S'[k, k-N] := [B'(k) | A'(k) B'(k-1) | \ldots | \\ A'(k, k-N+1) B'(k-N)]$$

Then Σ is called periodically smooth controllable at k_0 if $A(\cdot)$ and $B(\cdot)$ are bounded and if there exist constants ε, k_1 and N such that for all $k > 0$ there exists an integer $k_2(k)$ in the interval $(k_0 + (k-1) \times k_1, k_0 + k \times k_1)$ for which $S'[k_2 - N, k_2] S'^{\mathrm{T}}[k_2 - N, k_2] > \varepsilon I$.

For time-invariant systems we know that $S'[k, k-n]$ is full row rank at any time $k > k_0 + n$ (see Theorem 2). So by taking $k_1 = N = n$, $k_2 = k \times n$ and $\varepsilon = \sigma_{\min}$, where $\sigma_{\min}$ is the minimal singular value of $S'[k, k-n]$, we see that all time-invariant systems are periodically smooth controllable.

Theorem 4 *b*

Let Σ *be periodically smooth controllable at* k_0. Then $X_{\text{stab}}(k_0) = X^-(k_0)$. In other words: all periodically smooth controllable systems are smooth controllable.

Proof

Let x be an element of $X^-(k_0)$. Consider the state transformation from Corollary 3. If we apply this transformation on the system, we see that $x \in X^-(k_0)$ iff the second state component, $x_2'(k)$, converges to zero. This second component influences the first component of the transformed system via $A'_{12}(k)$ at any time k note that $A'_{12}(k)$ may be zero!

Now consider the interval (k_0, k_1) first. Let x_0 be the initial state of the system at k_0. The flow of the initial state at $k_2(1)$ is then $\bar{x}' := A(k_2(1), k_0) x_0$, and the sum of all components of the second state influencing the reachable subsystem by

$$e(k_2(1)) := \sum_{i=k_0}^{k_2(1)} A'_{11}(k_2(1), i) A'_{12}(i) x_2'(i)$$

Consider the input

$$u[k_0, k_2(1) - N - 1] = 0$$

and

$$u[k_2(1) - N, k_2(1)] = -S'^{\mathrm{T}}(S'S'^{\mathrm{T}})^{-1}(e(k_2(1)) + \bar{x}_1')$$

where $S' = S'[k_2(1), k_2(1) - N]$. With this input the reachable part of the state at time $k_2(1) + 1$, $x'(k_2(1) + 1)$, becomes zero. We show now by induction that it is possible to regulate $x_1'(k_2(k) + 1)$ to zero for any k. Therefore let t be any integer greater than one. Consider the interval $(k_0 + (t-2) \times k_1, k_0 + t \times k_1)$. The sum of all exogenous influences entering the reachable subsystem via matrix $A'(\cdot)$ from $k_2(t-1) + 1$ until $k_2(t)$ on is then

$$e(k_2(t)) := \sum_{i=k_2(t-1)+1}^{k_2(t)} A'_{11}(k_2(t), i) A'_{12}(i) x_2'(i)$$

Since by the induction hypothesis $x'(k_2(t)+1)$ is zero, application of the input

$$u[k_2(t-1)+1, k_2(t)-N-1]=0$$

and

$$u[k_2(t)-N, k_2(t)] = -S'^{\mathrm{T}}(S'S'^{\mathrm{T}})^{-1}e(k_2(t))$$

yields $x'_1(k_2(t)+1)=0$. Here $S'=S'[k_2(t), k_2(t)-N]$. So the induction argument is complete. Moreover, we observe that $\|u(k)\| \leqslant M\|e(k_2(t))\|$, since $S'S'^{\mathrm{T}} > \varepsilon I$. Therefore $\|x(k)\| \leqslant M'\|e(k_2(t))\|$ for all $k \in (k_2(t-1), k_2(t))$. Due to the convergence of $e(k_2(t))$ to zero when t tends to infinity, we conclude that $x(k)$ converges to zero when k tends to infinity if this input is applied which completes the proof. □

Another sufficient condition for smooth controllability is that the subsystem

$$x'_1(k+1) = A'_{11}(k)x'_1(k) + B'_1(k)u(k)$$

is uniformly stabilizable, in the sense given by Anderson and Moore (1981), for all $k > k_0$. However, even in the case where the system is stabilizable, this condition is not necessary either, as is seen by taking matrix $B(k) = \begin{bmatrix} 1/k \\ 0 \end{bmatrix}$ in Example 1.

We conclude this section by reconsidering Example 1. In this example

$$A'(k_0) = \begin{bmatrix} 2 & a \\ 0 & \frac{1}{2} \end{bmatrix}$$

where a is either $1/k$ or 0,

$$A'(k_0+1) = (0 \quad 1/2) \quad \text{and} \quad A'(k) = 1/2$$

for all $k > k_0 + 1$. Furthermore $B'_1(k) = 1$ for all k. So $S'[k, N]S'^{\mathrm{T}}[k, N] > 1$ for any k. Consequently the system is periodically smooth controllable. The potential stabilizability subspace is now found as $\{x \in \mathbb{R}^2 | (1/2)^k(0 \quad 1/2)x \to 0 \text{ if } k \text{ tends to infinity}\}$. This clearly equals $\mathbb{R}^2$. According to Theorem 4 b the stabilizability subspace is then equal to $\mathbb{R}^2$.

4. Target path controllability problems

In § 3 various problems were solved which all dealt with the question whether it was possible to regulate an initial state at some starting point to another predescribed state in time. However, this kind of dynamic controllability is unsatisfactory for the theory of economic policy. From the economic policy point of view the question whether it is possible to track any given target path over some time-interval from a prespecified point in time on is more interesting. This question has therefore gained a lot of attention in economics. Researchers who have been engaged in this subject include Tinbergen (1952), Preston and Pagan (1982), Aoki (1975), Aoki and Canzoneri (1979), and Wohltmann (1981). One contribution of this paper is that results are generalized to time-varying systems.

Following Wohltmann (1981) two questions are discussed: the length of the minimum policy lead (or reaction time) of the system, i.e. the minimum possible length of time needed to transfer the system from the initial state to the initial point of the desired target path; and the amount of time it is possible to keep the system on some desired target path once achieved. To answer these questions the notion of target path controllability is introduced.

Definition 5

Let p and q be positive integers. Then Σ is said to be target path controllable at k_0 with lead p and lag q if for any initial state $x(k_0)$ and for any reference output trajectory $y^*[k_0+p, k_0+p+q-1]$ there exists a control sequence $u[k_0, k_0+p+q-2]$ such that

$$y(k, k_0, x(k_0), u) = y^*(k)$$

for all

$$k_0 + p \leqslant k \leqslant k_0 + p + q - 1$$

In the sequel we abbreviate this property by TPC(k_0; p, q). In case the system is TPC(k_0; p, ∞) we say that the system is global target path controllable at k_0 with lead p.

Let $t = k + p$. The impulse response from $u[t+q-2, k]$ to $y[t+q-1, t]$:

$$\begin{bmatrix} C(t+q-1)\,B(t+q-2) & C(t+q-1)\,A(t+q-2)\,B(t+q-3) & \cdots & C(t+q-1)\,A[t+q-2, k+1]\,B(k) \\ 0 & \ddots & & \\ \vdots & & & \\ 0 \ldots 0 \quad C(t)\,B(t-1) & C(t)\,A(t-1)\,B(t-2) & \cdots & C(t)\,A[t-1, k+1]\,B(k) \end{bmatrix}$$

is denoted by $M(k; p, q)$. From this matrix a necessary and sufficient condition for target path controllability is easily derived. The result reads as follows.

Lemma 8

Σ is TPC(k_0; p, q) iff rank $M(k_0; p, q) = q \times r$.

Proof

Let t_0 be equal to $k_0 + p$. By definition Σ is target path controllable iff $y(k) = y^*(k)$ for all k satisfying $t_0 \leqslant k \leqslant t_0 + q - 1$. So the question is whether the following set of equations can be solved simultaneously for any $y^*[t_0, t_0+q-1]$

$$y^*(k+1) - C(k+1)\,A(k, k_0)\,x(k_0) = C(k+1)\,S[k, k_0]\,u[k, k_0], \qquad t_0 - 1 \leqslant k \leqslant t_0 + q - 2$$

This is possible iff the equation

$$y'[t_0 + q - 1, t] = M(k_0; p, q)\,u[t_0 + q - 2, k_0]$$

is solvable for any $y'[t_0, t_0 + q - 1]$. According to Lemma 2 this is equivalent to the requirement that the rank of matrix $M(k_0; p, q) = q \times r$. □

Since TPC(k_0; p, q) implies TPC(k_0; p, s) for any $0 < s \leqslant q$ the maximal obtainable lag q at time k_0 for a given lead p exists. From the rank condition of Lemma 8 we deduce immediately that the number of columns of matrix $M(k_0; p, q)$, $(p+q-1) \times m$, must be at least equal to $q \times r$. So for a given lead p an upper bound for the maximal obtainable lag q at k_0 results from the inequality

$$(p-1) \times m \geqslant q \times (r-m)$$

Conversely, one derives from this inequality for a predescribed lag q a lower bound for the lead needed to achieve TPC$(k_0; p, q)$. For Tinbergen models, i.e. $r \leqslant m$, we see that the upper bound and lower bound are respectively infinity and one. To determine for a given lead p the maximal obtainable lag q we must check the rank of matrix $M(k_0; p, s)$ for $s = 1, 2, \ldots$. We give a recursive algorithm to do this. The algorithm uses the following recursively defined subspaces.

Definition 6

Let $A(k)$, $B(k)$ and $S[k, k_0]$ be zero for $k < k_0$, and $t = p + k_0$. Then the subspace $\mathbb{R}_i(k_0, p)$ is recursively defined as:

$$\mathbb{R}_1(k_0, p) = \text{Im } S[t-2, k_0]$$

$$\mathbb{R}_i(k_0, p) = \text{Ker } C(t+i-2) \cap (A(t+i-3)\mathbb{R}_{i-1}(k_0, p) + \text{Im } B(t+i-3))$$

In the sequel the indices k_0 and p are omitted if it is clear from the context which indices are meant. The next lemma gives a geometric interpretation of these subspaces.

Lemma 9

$$\mathbb{R}_{q+1} = S[t+q-2, k_0] \text{ Ker } M(k_0; p, q), \ q > 0$$

Here $M(k_0; p, 0)$ is defined as 0.

Proof

The proof is carried out by induction on q. For $q = 0$ the equality is clear, due to the definition of $M(k_0; p, 0)$. Now assume that the statement holds for $i = q + 1$. Then by definition

$$\mathbb{R}_{q+2} = \text{Ker } C(t+q) \cap (A(t+q-1)\mathbb{R}_{q+1} + \text{Im } B(t+q-1))$$

Using the induction step and the definition of $S[k+q, q]$ this subspace can be rewritten as

$$\text{Ker } C(t+q) \cap S[t+q-1, k_0] \text{ Ker } [0 | M(k_0; p, q)]$$

Applying Lemma 6 yields that this subspace equals $S[t+q-1, k_0]$ Ker $M(k_0; p, q+1)$, which had to be proved. □

The algorithm which obtains the maximal lag for a given lead is derived from the next theorem by induction on the lag q.

Theorem 5

Let p be any positive integer, and $t = p + k_0$. Then Σ is TPC$(k_0; p, q)$ iff the following two conditions are satisfied for $i = t, \ldots, t+q-1$:

(a) $C(i)$ is full row rank; and

(b) $A(i-1)\mathbb{R}_{i+1} + \operatorname{Im} B(i-1) + \operatorname{Ker} C(i) = \mathbb{R}^n$.

Proof

By induction on q. Let $q = 1$. Then Σ is TPC(k_0; p, 1) iff $\operatorname{Im} C(t)S[t-1, k_0] = \mathbb{R}^r$. According to Corollary 2 this holds iff

$$\operatorname{Ker} C(t) + \operatorname{Im} S[t-1, k_0] = \mathbb{R}^n$$

and $C(t)$ is full row rank. Since $\operatorname{Im} S[t-1, k_0] = A(t-1)\mathbb{R}_1 + \operatorname{Im} B(t-1)$, this proves the first step.

Now assume that the theorem holds for $q-1$. In Lemma 8 we proved that Σ is TPC(k_0; p, q) iff $M(k_0; p, q)$ is full row rank. Now partition $M(k_0; p, q)$ as follows:

$$M(k_0; p, q) = \begin{bmatrix} C(t+q-1)B(t+q-2) & C(t+q-1)A(t+q-2)S[t+q-3, k_0] \\ 0 & M(k_0; p, q-1) \end{bmatrix}$$

Then according to Corollary 1 $M(k_0; p, q)$ is full row rank iff the following two conditions are satisfied:

(a) $\operatorname{Im} C(t+q-1)B(t+q-2) + C(t+q-1)A(t+q-2)S[t+q-3, k_0]$

$\times \operatorname{Ker} M(k_0; p, q-1) = \mathbb{R}^r$; and

(b) $M(k_0; p, q-1)$ is full row rank.

Now the second condition is satisfied by the induction hypothesis, while the first condition is equivalent to

$$C(t+q-1)S[t+q-2, k_0] \operatorname{Ker} [0 | M(k_0; p, q-1)] = \mathbb{R}^r.$$

As for $q = 1$ we use Corollary 2 to reformulate this as:

(a) $C(t+q-1)$ is full row rank; and

(b) $\operatorname{Ker} C(t+q-1) + S[t+q-2, k_0] \operatorname{Ker} [0 | M(k_0; p, q-1)] = \mathbb{R}^n$.

Since (b) can be rewritten as

$$\operatorname{Ker} C(t+q-1) + \operatorname{Im} B(t+q-2) + A(t+q-2)S[t+q-3, k_0] \operatorname{Ker} M(k_0; p, q-1) = \mathbb{R}^n$$

we can apply Lemma 9 to obtain the final condition

$$\operatorname{Ker} C(t+q-1) + \operatorname{Im} B(t+q-2) + A(t+q-2)\mathbb{R}_q = \mathbb{R}^n$$

Which completes the induction argument. □

Application of this theorem to the system

$$x(k+1) = \begin{bmatrix} A(k) & B(k) \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ I \end{bmatrix} v(k), \quad y(k) = (C(k)\,D(k))\,x(k)$$

with $v(k) = u(k+1)$ yields necessary and sufficient conditions for TPC$(k_0; p, q)$ for the system

$$x(k+1) = A(k)x(k) + B(k)u(k)$$
$$y(k) = C(k)x(k) + D(k)u(k)$$

This last system is studied extensively for time-invariant systems by Preston and Pagan (1982). As will be shown later on in this section their main conclusions concerning global TPC are easily rederived from Theorem 5.

Another special case is obtained when $C(i)$ is invertible at any point in time. This case has been studied by Wohltmann (1981) for time-invariant systems and with C equal to the identity matrix. In this paper he states that for this class of systems TPC with a lag greater than one can occur if the number of instrument variables is smaller than the number of output (=state) variables. He appears to illustrate this with an example. However, from Theorem 5 it is immediately clear that for systems with complete state observation, TPC$(k_0; p, q)$ for a lag greater than one occurs iff Im $B(i) = \mathbb{R}^n$ for $i = t, \ldots, t+q-1$. So the Tinbergen condition is also a necessary condition to get TPC for this class of systems. Wohltmann's statements concerning this subject in (Wohltmann 1981) are therefore incorrect.

Before considering the consequences of our algorithm for time-invariant systems we call attention to two algorithms developed for time-invariant systems which bear a great similarity with our algorithm. One of them is the algorithm developed by Willems (1981)—see also Willems *et al.* (1986) for an extension—to calculate the controllable L_∞-almost output nulling subspace. The difference between this algorithm and the one developed here is the initialization. Our algorithm in general does not start with $\mathbb{R}_1 = \{0\}$. Consequently the inclusion property of the algorithm, i.e. $\mathbb{R}_{i-1} \subset \mathbb{R}_i$, does not hold in our algorithm. Different subspaces will therefore be obtained ultimately.

The other algorithm is the algorithm developed in the same papers to calculate the controllable L_2-almost output nulling subspace. Though at a first glance our algorithm seems to be essentially different, the following definitions show the contrary. Define

$$S_1 = A(t-1)\mathbb{R}_1 + \text{Im } B(t-1)$$

and

$$S_{i+1} = A(t+i-2)\{S_i \cap \text{Ker } C(t+i-1)\} + \text{Im } B(t+i-2)$$

Then it is easily seen by induction that $S_i = A(t+i-2)\mathbb{R}_i + \text{Im } B(t+i-2)$. Therefore the equivalent condition resulting from Theorem 5 for TPC$(k_0; p, q)$ is that $C(t+i-1)S_i = \mathbb{R}^r$, for $i = 1, \ldots, q$. So again the conclusion can be drawn that the difference between both algorithms is only slight, namely the initialization. The advantage of the last algorithm is that the subspace S_i can be interpreted as: $S_i = \{x \in X \mid \exists w \in S_{i-1}, u \in U$ such that $A(t+i-2)w + B(t+i-2)u = x$ and $C(t+i-1)w = 0\}$—see e.g. Schumacher (1985).

For the derivation of results concerning global TPC the following lemma is important.

Lemma 10

Let Σ be time-invariant. If the lead p is greater than n, then for all $i \geq 2$ $\mathbb{R}_i \subset \mathbb{R}_{i-1}$.

Proof

The proof is carried out by induction on i. Let $i = 2$. By definition, $\mathbb{R}_2 = \operatorname{Ker} C \cap \operatorname{Im}[B|\ldots|A^{p-1}B]$. Due to the Cayley–Hamilton theorem this space equals: $\operatorname{Ker} C \cap \operatorname{Im}[B|\ldots|A^{p-2}B]$. Since the last subspace is contained in $\mathbb{R}_1$ this proves the lemma for the first step.

Now assume that the lemma holds for $i = k$. Then

$$\begin{aligned} \mathbb{R}_{k+1} &= \operatorname{Ker} C \cap (A\mathbb{R}_k + \operatorname{Im} B) && \text{(definition)} \\ &\subset \operatorname{Ker} C \cap (A\mathbb{R}_{k-1} + \operatorname{Im} B) && \text{(induction argument)} \\ &= \mathbb{R}_k && \text{(definition)} \end{aligned}$$

This completes the proof. □

From the second part of the proof of this lemma we see that if $\mathbb{R}_{k+1} = \mathbb{R}_k$ for some k then $\mathbb{R}_i = \mathbb{R}_k$ for any $i > k$. This property is used in the proof of Theorem 6. Theorem 6 states that a time-invariant system is global TPC with a lead $p > n$ iff the system is global TPC with lead $n + 1$. Since the Cayley–Hamilton theorem holds, the necessary and sufficient conditions derived in Theorem 5 for checking TPC$(0; n + 1, \cdot)$ reduce to one condition (apart from C being full row rank).

Theorem 6

Let Σ be time-invariant and $p > n$. Then Σ is global TPC with lead p iff the following hold:

(*a*) C is full row rank; and

(*b*) $A\mathbb{R}_{n+1} + \operatorname{Im} B + \operatorname{Ker} C = \mathbb{R}^n$.

Proof

Sufficient. From the remark above Theorem 6 and Lemma 10 we observe that either $\mathbb{R}_{n+1} = \mathbb{R}_n$ or $\mathbb{R}_{n+1} = 0$. Therefore the condition $A\mathbb{R}_{n+1} + \operatorname{Im} B + \operatorname{Ker} C = \mathbb{R}^n$ implies that

$$A\mathbb{R}_i + \operatorname{Im} B + \operatorname{Ker} C = \mathbb{R}^n$$

for any $i > n$. Since, due to our assumption on p, $\mathbb{R}_i$ is contained in $\mathbb{R}_{i-1}$ we see that the inclusion

$$A\mathbb{R}_i + \operatorname{Im} B + \operatorname{Ker} C \subset A\mathbb{R}_{i-1} + \operatorname{Im} B + \operatorname{Ker} C$$

holds. So the range condition

$$A\mathbb{R}_{n+1} + \operatorname{Im} B + \operatorname{Ker} C = \mathbb{R}^n$$

implies that the range of $A\mathbb{R}_i + \operatorname{Im} B + \operatorname{Ker} C$ equals $\mathbb{R}^n$ for any $i < n + 1$. Together with the assumption that the matrix C is full row rank we obtain now from Theorem 5 that Σ is TPC with lead p for any lag.

Necessary. This implication results as a special case from Theorem 5. □

A consequence of this theorem for time-invariant systems is that if Σ is not global TPC with lead $n + 1$, it will not be global TPC for any lead. For if the lead $p > n$ we see in Theorem 6 that TPC(0; p, $\cdot$) implies TPC(0; $n+1$, $\cdot$), while for a lead $p \leqslant n$ this implication is trivially satisfied. So the statement is proved with this by indirect demonstration.

A result derived in the proof of Theorem 6 which we should like to state separately is that TPC(0; p, $n+1$) implies TPC(0; p, k) for any $k > n$ if $p > n$.

5. Decoupled TPC problem

So far we considered the general target path controllability problem. A special case appears if, additionally to this problem we require that the ith input channel affects only the ith output channel, for $i = 1, \ldots, r$. We call this the decoupled TPC(k_0; p, q) problem. A proper definition is now given.

Definition 7

Σ is called decoupled TPC(k_0; p, q) if there exist compensators $F(k)$ and $G(k)$ such that with

$$u(k) = F(k)x(k) + G(k)w(k)$$

the impulse response on the interval $[k_0, k_0 + p + q]$ from $w_i \rightarrow y_j$ is zero for $i \neq j$, and moreover with this choice of control Σ is TPC(k_0; p, q).

Of course this problem only makes sense if the number of outputs equals the number of inputs. Note that the impulse response from $w \rightarrow y$ can always be made diagonal on the interval $[k_0 + p, k_0 + p + q]$, that is, choose

$$u[k_0, k_0 + p + q - 1] = M^{\mathrm{T}}(MM^{\mathrm{T}})^{-1} w[k_0 + p, k_0 + p + q]$$

where $M = M(k_0; p, q)$. However, in general this control does not make the impulse response on the whole interval $[k_0, k_0 + p + q]$ diagonal. To solve this problem, we have to consider the decoupling problem first.

The decoupling problem, as defined here, has been solved for time-invariant systems by Falb and Wolovich (1967). The first attempt to generalize this result to time-varying systems was taken by Porter (1969). In this paper a sufficient condition for the so-called integrator decoupling problem was given. This problem deals with the question under which conditions compensators $F(k)$ and $G(k)$ exist such that the closed-loop system $(Dy)(t) = \Lambda(t)w(t)t \in \mathbf{v}$ results. Here Λ is a diagonal matrix, and D is an operator which is defined below. This result has been generalized by Tzafestas and Pimenides (1976). Due to the decoupled inputs influencing the outputs succeeding $(Dy)(t)$ and which cannot be controlled by $u(t+1)$, a solution to the integrator decoupling problem does not in general solve the decoupling problem. We illustrate this with an example.

Example 2

Take

$$A(1) = I, \quad B^{\mathrm{T}}(1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad C(1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A(2) = I, \quad B^{\mathrm{T}}(2) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad C(2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A(3) = I, \quad B^{\mathrm{T}}(3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad C(3) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This system can be integrator decoupled, but not decoupled.

We now give a (strong) condition under which the decoupling problem is solvable. From this condition a sufficient condition for the solvability of the decoupled TPC problem is then immediately obtained. Before we state the result we introduce some notation.

In what follows, the ith row of matrix $C(k)$ is denoted by $c_i(k)$. Using this notation $a_i(k_0)$ is defined to be the minimum over all $k > k_0$ of the following set:

$$\{k | c_i(k) A(k-1, k_0+1) B(k_0) \neq 0\}$$

for $i = 1, \ldots, r$. Under the assumption that all $a_i(k_0)$ exist (as finite integers) we define $B^*(k_0)$ to be the matrix which has as its ith row entry the row $c_i(a_i)A(a_i - 1, k_0 + 1)B(k_0)$, for $i = 1, \ldots, r$. Finally the operator $(Dy)(k_0)$ is defined as:

$$\begin{aligned}(Dy)(k_0) &:= (d_1 y_1(k_0), \ldots, d_r y_r(k_0)) \\ &:= (y_1(a_1(k_0)), \ldots, y_r(a_r(k_0)))\end{aligned}$$

In the sequel the subscript k in $a_i(k)$ is dropped. Under the assumption that all $a_i(\cdot)$ exist the following propositions hold.

Proposition 1

Let $B^*(k)$ be invertible, and $d_i(k) = d_i$, for $i = 1, \ldots, r$. Then Σ can be decoupled.

Proof

Using the assumptions it is clear that Σ is also described by the following equations:

$$\begin{aligned}
x(k+1) &= A(k)x(k) + B(k)u(k), \quad x(k_0) = x \\
y_1(k) &= c_1(k) A(k-1, k_0)x, \quad k_0 < k < a_1(k_0) \\
&\vdots \\
y_r(k) &= c_r(k) A(k-1, k_0)x, \quad k_0 < k < a_r(k_0) \\
y_1(\alpha_1(k_0) + k - k_0) &= c_1(a_1) A(a_1 - 1, k+1) B(k)u(k) + c_1(a_1) A(a_1 - 1, k)x(k), \quad k \geqslant k_0 \\
&\vdots \\
y_r(\alpha_1(k_0) + k - k_0) &= c_r(a_r) A(a_r - 1, k+1) B(k)u(k) + c_r(a_r) A(a_r - 1, k)x(k), \quad k \geqslant k_0
\end{aligned}$$

Applying now the feedback $u(k) = B^{*-1}(k)w(k) - B^{*-1}(k)A^*(k)\,x(k)$ yields

$$y_i(k) = w_i(k), \quad k > a_i(k_0)$$

Here $A^*(k)$ is the matrix which has as its ith row entry the row $c_i(a_i)\,A(a_i - 1, k)$. Since $y_i(k)$ is also not influenced by $u(k)$ if $k_0 < k < a_i(k_0)$, this proves the theorem. □

From the theorem the following corollary is obvious.

Corollary 4

Σ is decoupled TPC(k_0; p, q) if the following hold:

(a) $d_i(k)$ is constant on the interval $[k_0; k_0 + p + q]$, for $i = 1, \ldots, r$;

(b) $B^*(k)$ is invertible on the interval $[k_0; k_0 + p + q]$; and

(c) $\max a_i(k_0) \leqslant p + k_0$.

Note that for time-invariant systems condition (a) in Corollary 4 is always satisfied and that the conditions (b) and (c) are then necessary too.

6. Strongly admissible reference trajectories

Strongly related to §§ 3 and 4 is the question whether a predescribed target path can be tracked exactly or at least ultimately. In the next two sections a characterization is given of all these trajectories.

A first, rather trivial, observation is that these trajectories are totally characterized by Σ. Though this seems to be a superfluous statement practice proves to be different. In economics, for example, conflicting goals often appear. Policy is then mostly designed to make the best of it in the short term. However, this policy can result in an unstable closed-loop system as has been shown by Engwerda and Otter (1986). So, despite the fact that a stabilizing policy may exist the intrinsic structure of the system is, in such situations, totally neglected in obtaining a short term optimal policy.

Important concepts that are used in the next two sections are now defined.

Definition 8

Let $-\infty < k_0 \leqslant k < l \leqslant \infty$, and x be the initial condition of Σ at k_0. A reference trajectory $y^*[k, l]$ is called strongly admissible for x at k_0 if $\exists u[k_0, l-1]$ such that

$$\| y[k, l] - y^*[k, l] \| = 0$$

and is called ϵ-almost strongly admissible for x at k_0 if $\exists u[k_0, l-1]$ such that

$$\| y[k, l] - y^*[k, l] \| < \varepsilon$$

A reference trajectory $y^*[k_0, \cdot]$ is called asymptotically admissible for x at k_0 if $\exists u[k_0, \cdot]$ such that $\| y(i) - y^*(i) \| \to 0$ for $i \to \infty$. The 'trivial' proposition then reads as follows.

Proposition 2

A reference trajectory $y^*[t, l+1]$ is strongly admissible for x iff there exists a $u^*[k_0, l]$ such that:

$$y^*(k) = C(k) x^*(k), \quad t \leqslant k \leqslant l + 1$$

$$x^*(k+1) = A(k) x^*(k) + B(k) u^*(k), \quad x^*(k_0) = x, \quad k_0 \leqslant k \leqslant l$$

Proof

The sufficiency of the condition is trivial. That the condition is also necessary is seen by the following reasoning. We know that

$$x(k+1) = A(k)x(k) + B(k)u(k), \quad x(k_0) = x$$
$$y(k) = C(k)x(k)$$

So the following equations hold for random $y^*(k)$:

$$x(k+1) = A(k)x(k) + B(k)u(k), \quad x(k_0) = x$$
$$y(k) - y^*(k) = C(k)x(k) - y^*(k)$$

Consider time t. Let $y^*(t)$ be strongly admissible. Consequently $y(t) - y^*(t)$ is zero. So $y^*(t) = C(t)x^*(t)$ for some $x^*(t)$ generated by the system. Since

$$x^*(t) = A(t-1, k_0)\, x + S[t-1, k_0]u[t-1, k_0]$$

we conclude that there exists a $u^*[k_0, t-1]$ such that $x(t) = x^*(t)$. By induction on k it is now easily verified that the relation holds for any $t \leqslant k \leqslant l$. □

Input–output descriptions are sometimes easier to perform calculations with than state space descriptions. For time-invariant systems it is always possible to derive from a state space representation an input–output representation. This is among other things due to the Cayley–Hamilton theorem. This property ceases to hold for time-varying systems. The attempt to generalize the Cayley–Hamilton theorem by assuming that $A(k_0 + N, k_0)$ is a linear function of $A(k_0), \ldots, A(k_0 + N - 1, k_0)$ for any $N > N_0$ only makes sense if matrix $C(k)$ is time-invariant; so the problem is not trivial. We give here a sufficient condition for the existence of such a relationship. This condition is reconstructibility. This concept is defined e.g. by Ludyck (1981, Chap. 2.4). In this chapter he also gives a necessary and sufficient condition for reconstructibility. However, the proof of this conjecture is incorrect. Therefore we provide a correct proof. The proof stems from a proof that Willems (1980) gives for the reconstructibility of time-invariant systems.

Definition 9

A state is reconstructible at k_0 if there exists a time $k_0 - N$ such that $x(k_0)$ is uniquely determined by $u[k_0 - N, k_0 - 1]$ and $y[k_0 - N, k_0 - 1]$; Σ is called i/o-convertible at k_0 if there exists an $N > 0$ and matrices $P_k(i)$, $Q_k(i)$ such that for all $k > k_0$ Σ is described by the input–output relation

$$y(k) = \sum_{i=k_0-N}^{k-1} P_k(i)\,y(i) + Q_k(i)\,u(i)$$

Proposition 3

Σ is reconstructible at k_0 iff there exists a positive integer N such that

$$\operatorname{Ker} W^{\mathrm{T}}[k_0 - N, k_0 - 1] \subset \operatorname{Ker} A(k_0 - 1, k_0 - N)$$

—for the definition of W see § 2.

Proof

First note that

$$x(k_0) = A(k_0 - 1, k_0 - N)\,x(k_0 - N) + S[k_0 - 1, k_0 - N]u[k_0 - 1, k_0 - N]$$

and

$$y(k) = C(k)A(k-1, k_0 - N)x(k_0 - N) + C(k)S[k-1, k_0 - N]u[k-1, k_0 - N]$$
$$k = k_0 - N + 1, \ldots, k_0 - 1$$

By definition Σ is reconstructible at k_0 iff from the past observations

$$v(k_0 - N) := C(k_0 - N)x(k_0 - N)$$

and

$$v(k+1) := C(k+1)A(k, k_0 - N)x(k_0 - N)$$

for $k = k_0 - N, \ldots, k_0 - 2$, and $A(k_0 - 1, k_0 - N)x(k_0 - N)$ can be determined uniquely. This is not possible for all states at k_0 iff there exist two states $x'(k_0 - N)$ and $x(k_0 - N)$ such that

$$C(k_0 - N)x'(k_0 - N) = C(k_0 - N)x(k_0 - N)$$
$$C(k+1)A(k, k_0 - N)x'(k_0 - N) = C(k+1) \times A(k, k_0 - N)x(k_0 - N) \quad \text{for } k = k_0 - N, \ldots, k_0 - 2$$

and

$$A(k_0 - 1, k_0 - N)x'(k_0 - N) \neq A(k_0 - 1, k_0 - N)x(k_0 - N)$$

or equivalently

$$x'(k_0 - N) - x(k_0 - N) \in \operatorname{Ker} W^{\mathrm{T}}[k_0 - N, k_0 - 1]$$

and

$$x'(k_0 - N) - x(k_0 - N) \notin \operatorname{Ker} A(k_0 - 1, k_0 - N) \qquad \square$$

Since $A(k_0 - 1, k_0 - N)x(k_0 - N)$ is uniquely determined by $v(k)$, $k_0 - N \leqslant k \leqslant k_0 - 1$, we can rewrite $A(k_0 - 1, k_0 - N)x(k_0 - N)$ as $[X(k_0 - N)|\ldots|X(k_0 - 1)]v[k_0 - N, k_0 - 1]$ for some matrix $[X(k_0 - N)|\ldots|X(k_0 - 1)]$. A direct consequence is the Proposition 4.

Proposition 4

Σ is i/o convertible at k_0 if Σ is reconstructible at k_0. The input–output relation is given by

$$y(k) = C(k)A(k-1, k_0)\{[X(k_0 - N)|\ldots|X(k_0 - 1)] \times v[k_0 - N, k_0 - 1] + S[k_0 - 1, k_0 - N]u[k_0 - 1, k_0 - N]\} + C(k)S[k-1, k_0]u[k-1, k_0]$$

Since this input–output relation depends severely on the initial state $x(k_0)$ of the system we give sufficient conditions under which a dynamic input–output relation (i.e. a relation which is independent of the initial state of the system) is obtained. This provides the contents of Theorem 7.

Theorem 7

Assume that there exists an $N > 0$ such that $W[k, k+N]$ is full row rank for any k.

Then the following input–output relation holds for Σ.

$$y(k+N+1) = C(k+N+1)\{W[k, k+N]W^{\mathrm{T}}[k, k+N]\}^{-1}W[k, k+N] \times \left(y[k, k+N] - \begin{bmatrix} 0 \\ M(k; 1, N)u[k+N-1, k] \end{bmatrix} \right)$$

Proof

The proof is obvious from the consideration that

$$y[k, N+k] = W^{\mathrm{T}}[k, N+k]x(k) + \begin{bmatrix} 0 \\ M(k; 1, N-1)u[k+N-1, k] \end{bmatrix}$$

for any k. □

Note that

$$W[k+1, N+k+1]W^{\mathrm{T}}[k+1, N+k+1] = W[k, N+k]W^{\mathrm{T}}[k, N+k] + A^{\mathrm{T}}(k+N-1, k)C^{\mathrm{T}}(k+N)C(k+N)A(k+N-1, k) - A^{\mathrm{T}}(k)C^{\mathrm{T}}(k+1)C(k+1)A(k)$$

From this equality we obtain easily a recursion formula for the calculation of

$$\{W[k, k+N]W^{\mathrm{T}}[k, k+N]\}^{-1}$$

In the previous part of this section we characterized all strongly admissible trajectories. However, the question of how the strong admissibility of a prespecified reference trajectory can immediately be checked was not answered. Moreover, the question whether there exists an ε such that the trajectory is ε-almost strongly admissible when it is not strong admissible arises. These questions are answered in the final theorems of this section.

Theorem 8

A reference trajectory $y^*[k_0+p, k_0+p+q-1]$ is strongly admissible at k_0 iff

$$\operatorname{rank}\,[M(k_0; p, q)|z[k_0+p, k_0+p+q-1]] = \operatorname{rank} M(k_0; p, q)$$

Here $z(i) = y^*(i) - C(i)A(i-1, k_0)x(k_0)$

Proof

The proof is obvious from Lemma 2.

Theorem 9

A reference trajectory $y^*[k_0+p, k_0+p+q-1]$ is ε-almost strongly admissible iff

$$\|(M(k_0; p, q)M^{+}(k_0; p, q) - I)z[k_0+p, k_0+p+q-1]\| < \varepsilon \qquad (2)$$

Here $M^{+}(k_0; p, q)$ is the Moore–Penrose inverse of $M(k_0; p, q)$—see Lancaster and Tismenetsky (1985, Chap. 12.8)—and z is as defined in Theorem 8.

Proof

Using the definition of z from Theorem 8 we write $z[k_0 + p, k_0 + p + q - 1]$ as $M(k_0; p, q) u[k_0, k_0 + p + q - 2] + y^*[k_0 + p, k_0 + p + q - 1] - y[k_0 + p, k_0 + p + q - 1]$. So $\| y[k_0 + p, k_0 + p + q - 1] - y^*[k_0 + p, k_0 + p + q - 1] \| < \varepsilon$ iff

$$\| z[k_0 + p, k_0 + p + q - 1] - M(k_0; p, q) u[k_0, k_0 + p + q - 2] \| < \varepsilon$$

However, $\min_u \| M(k_0; p, q) u[k_0, k_0 + p + q - 2] - z[k_0 + p, k_0 + p + q - 1] \|$ is obtained by the least-squares approximation

$$u[k_0, k_0 + p + q - 2] = M^+(k_0; p, q) z[k_0 + p, k_0 + p + q - 1]$$

(see Lancaster (1985, p. 436). This proves the result. □

From Theorem 9 it is clear that for a given lead p the minimal ε for which a trajectory is ε-almost strongly admissible exists. This ε is given by (2) where the inequality has to be replaced by an equality.

7. Asymptotically admissible reference trajectories

In § 5 we derived a criterion to check whether a certain reference trajectory could be tracked exactly or not. Moreover an exact characterization was given of how a reference trajectory has to be generated in order to be strongly admissible. We shall now treat the problem of tracking a reference trajectory asymptotically. To tackle this problem, we assume that the input is chosen as a mixture of static/dynamic, state/output feedback, that is

$$u(k) = E(k) w(k) + F(k) x(k) + H(k) z(k) + D(k) y(k) + g(k)$$

where

$$w(k+1) = M(k) w(k) + N(k) x(k)$$

$$z(k+1) = P(k) z(k) + Q(k) y(k)$$

Then, for random $u^*(k)$, $u^*(k-1)$, $w^*(k+1)$, $w^*(k)$, $z^*(k+1)$, $z^*(k)$, $x^*(k+1)$, $x^*(k)$, $y^*(k)$, $y^*(k+1)$ the following closed-loop system results:

$$\begin{bmatrix} I & 0 & 0 & -B(k) & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & -Q(k) \\ 0 & 0 & 0 & I & -D(k) \\ 0 & 0 & 0 & 0 & I \end{bmatrix} e(k+1) = \begin{bmatrix} A(k) & 0 & 0 & 0 & 0 \\ N(k) & M(k) & 0 & 0 & 0 \\ 0 & 0 & P(k) & 0 & 0 \\ F(k) & E(k) & H(k) & 0 & 0 \\ C(k) & 0 & 0 & 0 & 0 \end{bmatrix} e(k)$$

$$+ \begin{bmatrix} A(k) x^*(k) + B(k) u^*(k) - x^*(k+1) \\ M(k) w^*(k) + N(k) x^*(k) - w^*(k+1) \\ P(k) z^*(k) + Q(k) y^*(k) - z^*(k+1) \\ F(k) x^*(k) + E(k) w^*(k) + H(k) z^*(k) + g(k) \\ C(k) x^*(k) - y^*(k) \end{bmatrix}$$

where

$$e^{\mathrm{T}}(k+1) = [(x(k+1) - x^*(k+1))^{\mathrm{T}}, (w(k+1) - w^*(k+1))^{\mathrm{T}}, (z(k+1) - z^*(k+1))^{\mathrm{T}}, (u(k) - u^*(k))^{\mathrm{T}}, (y(k) - y^*(k))^{\mathrm{T}}]$$

This error equation can be rewritten as

$$S(k)e(k+1) = A'(k)e(k) + v(k)$$

To give some more insight into the properties of an admissible trajectory the next theorem, which immediately results from Lemma 1, is stated.

Theorem 10

An admissible reference trajectory is generated as follows:

$$x^*(k+1) = A(k)x^*(k) + B(k)u^*(k) + v_1(k)$$

$$y^*(k) = C(k)x^*(k) + v_2(k)$$

with $v_i(k) \to 0$ when k tends to infinity.

This condition is also sufficient if the input stabilizes the system. Using Theorem 7, Theorem 10 can be reformulated in the following way: a necessary condition for a reference trajectory to be admissible is that it is generated in the limit by the same input–output recurrence relation as the system Σ. The exact characterization for admissibility of a reference trajectory reads as follows.

Theorem 11

A reference trajectory is admissible at k_0 iff there exist $v_i[k_0, \cdot]$, for $i = 1, \ldots, 5$, such that the following conditions are met:

(*a*) $$e(k_0) := [(x(k_0) - x^*(k_0))^{\mathrm{T}}, (w(k_0) - w^*(k_0))^{\mathrm{T}}, (z(k_0) - z^*(k_0))^{\mathrm{T}}]$$

is stabilized by means of $v_i[k_0, \cdot]$, for $i = 1, \ldots, 5$, in the linear system

$$e(k+1) = \begin{bmatrix} A(k) + B(k)F(k) & B(k)E(k) & B(k)H(k) \\ N(k) & M(k) & 0 \\ Q(k)C(k) & 0 & P(k) \end{bmatrix} e(k) + \begin{bmatrix} v_1(k) + B(k)v_4(k) \\ v_2(k) \\ v_3(k) + Q(k)v_5(k) \end{bmatrix}$$

(*b*)
$$x^*(k+1) = A(k)x^*(k) + B(k)u^*(k) + v_1(k)$$

$$w^*(k+1) = M(k)w^*(k) + N(k)x^*(k) + v_2(k)$$

$$z^*(k+1) = P(k)z^*(k) + Q(k)y^*(k) + v_3(k)$$

$$g(k) = -F(k)x^*(k) - E(k)w^*(k) - H(k)z^*(k) + v_4(k)$$

$$y^*(k) = C(k)x^*(k) + v_5(k)$$

where $v_i(k) \to 0$ if k tends to infinity.

Proof

Sufficient. From Lemma 1 we know that it is necessary for the convergence of $e(\cdot)$ in

$$S(k)e(k+1) = A'(k)e(k) + v(k)$$

that $S^{-1}(k)v(k)$ converges to zero. Consequently $v(k)$ has to converge to zero. This implies the second condition in the theorem. Straightforward multiplication shows that $S^{-1}(k)A'(k)$ equals $\begin{bmatrix} T(k) & 0 \\ W(k) & 0 \end{bmatrix}$ for some matrices T and W. Using this structure of the error equation matrix, the first condition of the theorem now immediately results too.

Necessary. By making use again of the structure of the error equation matrix as derived above this part is easily proved. □

We conclude this section with an illustration of Theorem 10. In the work by Engwerda (1986), the infinite time quadratic tracking problem has been solved under some conditions. As a special case the following problem has been treated:

$$\min_{u[0,\cdot]} \lim_{N\to\infty} \sum_{k=0}^{N-1} \{(y(k)-y^*(k))^{\mathrm{T}} Q(y(k)-y^*(k)) + (u(k)-u^*(k))^{\mathrm{T}} R(u(k)-u^*(k))\}$$
$$+ (y(N)-y^*(N))^{\mathrm{T}} Q(y(N)-y^*(N))$$

where Q and R are positive definite symmetric matrices and $y(k+1)$ is given by the system equation

$$y(k+1) = Ay(k) + Bu(k) + Gd(k)$$

Here $Gd(k)$ is a known exogenous noise component. The optimal solution to this problem is

$$u(i) = -(R + B^{\mathrm{T}}KB)^{-1}B^{\mathrm{T}}(KAy(i) + KGd(i) - h(i+1) - B(B^{\mathrm{T}}B)^{-1}Ru^*(i))$$

where K is the positive definite solution of the algebraic Riccati equation

$$K = A^{\mathrm{T}}\{K - KB(R + B^{\mathrm{T}}KB)^{-1}B^{\mathrm{T}}K\}A + Q$$

and $h(i)$ is given by the recurrence equation

$$h(1) = \sum_{k=1}^{\infty} \{(A-BH)^{\mathrm{T}}\}^{k-1}\{Qy^*(k) - (RH)^{\mathrm{T}}u^*(k) - (A-BH)^{\mathrm{T}}KGd(k)\}$$

$$h(i+1) = \{(A-BH)^{\mathrm{T}}\}^{-1}\{h(i) - Qy^*(i) + (RH)^{\mathrm{T}}u^*(i)\} + KGd(i)$$

Here H denotes the matrix $(R + B^{\mathrm{T}}KB)^{-1}B^{\mathrm{T}}KA$. In this paper it has been proved that this control stabilizes the closed-loop system. So, application of Theorem 10 now yields that the error $[(y(k)-y^*(k))^{\mathrm{T}}, (u(k)-u^*(k))^{\mathrm{T}}]$ converges to zero if and only if the following vector converges to zero when k tends to infinity:

$$\begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix}$$
$$= \begin{bmatrix} Ay^*(k) + Bu^*(k) + Gd(k) - y^*(k+1) \\ (R + B^{\mathrm{T}}KB)^{-1}B^{\mathrm{T}}(h(k+1) - KAy^*(k) - KGd(k) + B(B^{\mathrm{T}}B)^{-1}Ru^*(k)) - u^*(k) \end{bmatrix}$$

Now substitution of $v_1(k)$ into $v_2(k)$ yields

$$v_2(k) = -(R + B^{\mathrm{T}}KB)^{-1}B^{\mathrm{T}}(Ky^*(k+1) - h(k+1) - v_1(k)).$$

So we conclude that $v_2(k)$ converges to zero if and only if $B^{\mathrm{T}}(Ky^*(k+1) - h(k+1))$ converges to zero.

Summarizing, we have the following result.

Theorem 12

A reference trajectory is admissible in the infinite time quadratic tracking problem iff the following two conditions hold for the reference trajectory:

(*a*) $y^*(k+1) - Ay^*(k) - Bu^*(k) - Gd(k) \to 0$ when $k \to \infty$; and

(*b*) $B^{\mathrm{T}}(Ky^*(k+1) - h(k+1)) \to 0$ when $k \to \infty$.

8. Conclusions

In this paper we have treated various target point and target path problems for time-varying linear discrete time systems. For some of the target point problems new elementary proofs have been given, while for the output-controllability and stabilizability problem new results have been obtained. For the general target path controllability problem a geometric recursive algorithm has been derived to check the solvability of the problem. This algorithm appears to bear a great similarity to other algorithms derived in the literature to calculate controllable almost output nulling subspaces for time-invariant systems. An advantage of this algorithm is that easily (existing) results for time-invariant systems concerning the TPC problem are re-obtained. For a special problem, the decoupled TPC problem, a sufficient condition has been given for solvability.

Finally, a characterization has been given of the trajectories which can (almost or ultimately) be tracked. A sufficient condition has been provided for obtaining a dynamic input–output relation from the state space representation. Furthermore, a method to check the (almost) strong admissibility of a prespecified reference trajectory has been given.

References

Albrecht, F., Grasse, K. A., and Wax, N., 1986, *I.E.E.E. Trans autom. Control*, **31,** 569.

Anderson, B. D. O., and Moore, J. B., 1981, *SIAM J. Control Optim.*, **19,** 20.

Aoki, M., 1975, *Rev. econ. Stud.*, **42,** 293.

Aoki, M., and Canzoneri, M., 1979, *Int. econ. Rev.*, **30,** 605.

Brockett, R. W., and Mesarovic, M. D., 1965, *J. math. Analysis Applic.*, **11,** 548.

Engwerda, J. C., 1986, *Memorandum Cosor 86-08*, Eindhoven University of Technology.

Engwerda, J. C., and Otter, P. W., 1986, *Memorandum Cosor 86-20*, Eindhoven University of Technology.

Falb, P. L., and Wolovich, W. A., 1967, *I.E.E.E. Trans. autom. Control*, **12,** 651.

Grasse, K. A., 1986, *I.E.E.E. Trans. autom. Control*, **31,** 571.

Kalman, R. E., Falb, P. L., and Arbib, M. A., 1969, *Topics in Mathematical System Theory* (New York: McGraw-Hill).

Kwakernaak, H., and Sivan, R., 1972, *Linear Optimal Control Systems* (New York: Wiley).

Lancaster, P., and Tismenetsky, M., 1985, *The Theory of Matrices*, second edition (London: Academic Press).

Ludyck, G., 1981, *Time-Variant Discrete-Time Systems; Advances in Control Systems and Signal Processing*, Vol. 3 (Braunschweig, FRG: Vieweg).

MEDITCH, J. S., 1969, *Stochastic Optimal Linear Estimation and Control* (New York: McGraw-Hill).
PORTER, W. A., 1969, *I.E.E.E. Trans. autom. Control*, **14,** 378.
PRESTON, A. J., 1974, *Rev. Econ. Stud.*, **41,** 65.
PRESTON, A. J., and PAGAN, A. R., 1982, *The Theory of Economic Policy* (New York: Cambridge University Press).
PRESTON, A. J., and SIEPER, E., 1977, *Applications of Control Theory to Economic Analysis*, edited by J. D. Pitchford and S. J. Turnovsky (Amsterdam: North Holland), Essay 9, pp. 215–255.
SAIN, M. K., and MASSEY, J. L., 1969, *I.E.E.E. Trans. autom. Control*, **14,** 141.
SCHUMACHER, J. M., 1985, *I.E.E.E. Trans. autom. Control*, **30,** 1075.
TINBERGEN, J., 1952, *On the Theory of Economic Policy* (Amsterdam: North Holland).
TZAFESTAS, S. G., and PIMENIDES, T. G., 1976, *I.E.E.E. Trans. autom. Control*, **21,** 410.
WILLEMS, J. C., 1981, *I.E.E.E. Trans. autom. Control*, **26,** 235; 1980, Lecture notes on Systems Theory, University of Groningen.
WILLEMS, J. C., KITAPÇI, A., and SILVERMAN, L. M., 1986. *SIAM J. Control Optim.*, **24,** 323.
WOHLTMANN, H. W., 1981, *J. Econ.*, **41,** 39; 1985, *I.E.E.E. Trans. autom. Control*, **30,** 84.